

Theoretical bounds on the tensor-to-scalar ratio.

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Tensor modes in the cosmic microwave background are one of the most robust signatures of inflation. We derive theoretical bounds on the tensor fraction, as a generalization of the well-known Lyth bound. We comment on a previously derived generalization, the so-called Efstathiou-Mack relationship. We also derive a new absolute upper bound on tensors using de Sitter entropy bounds.

Understanding the initial conditions that led to structure formation in our universe is one of the most important issues in modern cosmology. Among the variety of available scenarios, inflation occupies a special place. In addition of being a theoretically attractive paradigm, it is compatible with all the available observational data. In its simplest version, single field slow-roll inflation is realized through a canonically normalized scalar field minimally coupled to Einstein gravity. To match with observations, the height and the slope of the potential have to obey special relationships. However, in most of the models, the excursions of the scalar field (inflaton) is at least of order of the reduced Planck scale $M_P = (8\pi G_N)^{-1/2}$. This raises the concern whether effective field theory used to derive the predictions are still valid. This is especially important considering the fact that quantum gravity corrections are likely to spoil the delicate balance between the height of the potential and its slope once the inflaton is allowed to travel over Planckian distances in field space*. Inflation, in this sense, is unique because it is doubly UV sensitive: firstly because of the mass of the inflaton owing to its scalar nature, and secondly because of the unavoidable proliferation of dangerous UV-suppressed operators. Of course, supersymmetry addresses the first issue, however the fine-tuning of the infinite tower of Planck-suppressed operators inevitably remains.

On the other hand, inflation predicts a stochastic background of gravitational waves (GWs) which magnitude is related to the energy scale at which inflation happened [1] and, more importantly, the excursion of the inflaton field [2]. According to the *Lyth bound* [2], positive detection of tensors would mean super-Planckian values of the inflaton which is clearly interesting both from the model building standpoint and the observational one. It would also imply serious conceptual re-thinking about effective field theory. Given the fact that inflation offers an unequal and unique opportunity to access the Planck scale, and in the absence of experimental guidance on that question, it is important to use theoretical

consistency to make some progress. In this article, we scrutinize this issue and derive theoretical bounds on the tensor fraction.

We begin by recalling the necessary conditions under which the bounds hold. First, we assume that gravity is described by Einstein general relativity (GR) and that inflation is of the slow-roll variety defined by the usual flatness conditions

$$\epsilon \equiv \frac{1}{2} M_P^2 |V'/V|^2 \ll 1 \text{ and } |\eta| \equiv M_P^2 |V''/V| \ll 1, \quad (1)$$

together with the slow-roll approximations $3H\dot{\phi} \simeq -V'(\phi)$ and $3H^2 M_P^2 \simeq V(\phi)$, where H is the Hubble rate. Second, we assume that the primordial curvature perturbation ζ , which is constant on super-horizon scales, is produced through the vacuum fluctuation of one or more light scalar fields. Using this approximation we can compute the spectrum of curvature perturbation produced by the inflaton $\mathcal{P}_\zeta(k)^{1/2} = H^2/2\pi\dot{\phi}$. In the case of a single field, that we assume from now on, the later quantity should be equated with the observed value $\sim 5 \times 10^{-5}$. The scalar index n_s is given in the slow-roll approximation by $n_s = 1 + 2\eta_* - 6\epsilon_*$. The latest WMAP 7 yrs dataset [3]

$$n_s = 0.963 \pm 0.012 \quad 68\% \text{ CL} \quad (2)$$

excludes the Harrison-Zeldovich-Peebles spectrum by more than $3\text{-}\sigma$ and strongly favors a red tilt. On the other hand the tensor-to-scalar ratio $r \equiv 16\epsilon_*$ is subject to the bound $r < 0.24$ at 95% CL (WMAP+BAO+ H_0). As we will see, these bounds are already quite constraining. For instance, models based on $V \propto \phi^p$, with $p \geq 4$ are ruled-out by this bound at least at the $3\text{-}\sigma$ level. Moreover, hybrid models based on $V(\phi) = V_0 + m^2 \phi^2/2$, with $m^2 > 0$, are also ruled-out because of their blue-tilted spectrum. Therefore, we will focus hereafter on chaotic and hilltop models.

* Notice that in addition to spoiling the flatness of the potential, non-renormalizable operators $\lambda_n \phi^{n+4}/M_P^n$ (with $n \geq 1$) will make the energy density $\gg M_P^4$ once the inflaton takes super-Planckian values.

[†] Here we denote quantities at horizon exit with a $*$ in the subscript.

[‡] This bound becomes a bit stronger $r < 0.20$ (95% CL) using WMAP+BAO+SN data, however as it does not include systematic errors in SN data, we will not consider it.

Our starting point to derive the bounds is the definition of the classical [§] number of e-folds

$$dN = M_P^{-1} d\phi / \sqrt{2\epsilon(\phi)}, \quad (3)$$

The total number of e-folds is obtained as usual by integrating Eq. (3) starting from horizon exit to the end of inflation. The original bound [2] was derived by integrating Eq. (3) for modes corresponding to the multipoles $2 < \ell \lesssim 100$. The crucial point is that ϵ and thus r does not change too much during the last $\Delta N \simeq 4$ e-folds corresponding to these modes. Thus one can integrate Eq. (3) and obtain the relationship

$$\frac{\Delta\phi_4}{M_P} \simeq \left(\frac{r}{0.52}\right)^{1/2} \simeq 0.14 \left(\frac{r}{0.01}\right)^{1/2}, \quad (4)$$

where $\Delta\phi_4$ is the inflaton displacement during the last 4 e-folds. However, if one wants to extend that bound to the whole N e-folds, one cannot assume anymore negligible variation of the slow-roll parameters. In the following, we will derive bounds on the inflaton field excursion taking into account the variation of ϵ during inflation. Before doing so, we will derive a general inequality which is valid for all slow-roll models. Using the fact that the number of e-folds N is just the area under the curve $(1/M_P \sqrt{2\epsilon(\phi)})$ between ϕ_* and ϕ_{end} , we can write that [¶]

$$\frac{\Delta\phi}{M_P} \frac{1}{\sqrt{2\epsilon_{\text{max}}}} \leq N \leq \frac{\Delta\phi}{M_P} \frac{1}{\sqrt{2\epsilon_{\text{min}}}}, \quad (5)$$

where $\Delta\phi \equiv |\phi_{\text{end}} - \phi_*|$ is the total inflaton excursion during N e-folds and ϵ_{min} and ϵ_{max} are the minimum and maximum values of ϵ during that period. Eq. (5) is the first main result of this paper. In principle, ϵ_{min} could be vanishingly small, however for the validity of the semiclassical description, the smallest possible value, which corresponds to the phase transition to the eternal inflation regime [4], is $\epsilon_c \equiv 3/4\pi^2 (H/M_P)^2$. Plugging this value, we obtain $N \lesssim \Delta\phi/H$. Typically though, the slow-roll parameter grows monotonically during inflation from ϵ_* until the breakdown of slow-roll $\epsilon_{\text{end}} \sim 1$. This is in fact the case for hilltop and chaotic inflation scenarios to which we will apply the bounds Eq. (5).

Let us begin with hilltop inflation models where ϵ increases monotonically as the universe inflates. Using the right-hand side of Eq. (5), we get a bound that can be written in terms of the tensor-to-scalar ratio as

$$r < 0.002 \left(\frac{\Delta\phi}{M_P}\right)^2 \left(\frac{60}{N}\right)^2, \quad (6)$$

[§] By classical number of e-folds, we mean N in the slow-roll non-eternal inflation regime.

[¶] We are using the following basic property of definite integrals: if a function $f(x)$ is bounded on an interval $a \leq x \leq b$ i.e. $A \leq f(x) \leq B$ then $(b-a)A \leq \int_a^b dx f(x) \leq (b-a)B$.

Now, let us focus on quadratic hilltop models which potential is given by $V(\phi) = V_0 - m^2 \phi^2 / 2 + \dots$ with $m^2 > 0$. The dots stand for higher order terms that will make the potential bounded from below and which might dominate after horizon exit. In this case, the tensor-to-scalar ratio is given by

$$r = 2(1 - n_s)^2 \left(\frac{\phi_{\text{end}}}{M_P}\right)^2 e^{N(1-n_s)}. \quad (7)$$

From Eq. (7), we can use the WMAP 7 yrs bound on the scalar spectral tilt $1 - n_s \lesssim 0.04$ to derive a more stringent bound on r which reads [5] ^{**}

$$r \lesssim 0.0003 \left(\frac{\Delta\phi}{M_P}\right)^2 \left(\frac{60}{N}\right)^2, \quad (8)$$

which is one order of magnitude stronger than the previous bound^{††}.

One can also consider variants of the hilltop scenario where higher powers of ϕ dominate at the top of the potential. If there is no $\phi \rightarrow -\phi$ symmetry then the first term in the potential will be cubic. However, and in addition to the fact that the potential is not bounded from below, this leads to a spectral index on the verge of the 3 standard deviations of the WMAP allowed region. If on the other hand there is a symmetry $\phi \rightarrow -\phi$ and the quadratic term is, for some reason, negligible^{‡‡}, then the potential will be $V(\phi) = V_0 - \lambda\phi^4/4$. If the inflation is responsible for the generation of density perturbations then $\lambda \simeq 10^{-12}$ regardless of the value of V_0 . This tiny value of the coupling can be justified if one think of ϕ as a modulus described by the potential $V_0(1 - \lambda_4\phi^4/M_P^4)$ with $\lambda_4 \sim 1$ and $V_0 \sim 10^{15} \text{ GeV}$. In the following we will consider general values of V_0 , allowing both sufficient reheating and the use of standard QFT and GR methods $\text{MeV}^4 \ll V_0 \ll M_P^4$. The bound Eq. (6) still applies in this case, however one can derive a tighter relationship between $\Delta\phi$ and r which can be written

$$\frac{\Delta\phi}{m_P} = \frac{N^{3/4}}{2\sqrt{\pi}} r^{1/4}, \quad (10)$$

where here $m_P = G_N^{-1/2}$ is the Planck mass. For $N = 60$, Eq. (10) gives indeed the Efstathiou-Mack relationship $\Delta\phi/m_P \approx 6r^{1/4}$ which was derived empirically in [6].

^{**} One can use the property that $a^n e^{-a} < b^n e^{-b}$ for any $a > b > n \geq 0$.

^{††} If the potential does not steepen after horizon exit, the tensor-to-scalar ratio will satisfy a more relaxed bound [2]

$$r = 8(1 - n_s) e^{N(1-n_s)} \lesssim 0.03 (60/N). \quad (9)$$

^{‡‡} Notice that, apart from shift symmetries $\phi \rightarrow \phi + c$, there is no symmetry that can consistently forbid the quadratic term.

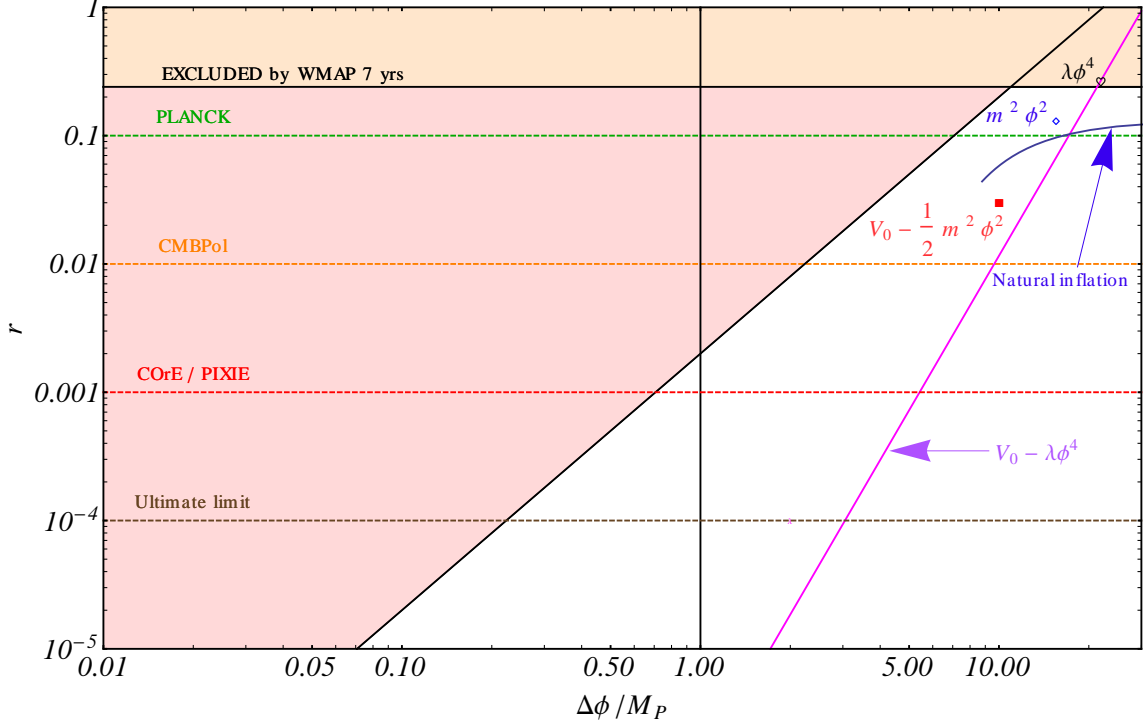


FIG. 1: The tensor-to-scalar ratio versus field excursion for inflationary potentials compatible with WMAP 7 yrs result on the spectral tilt Eq. (2). The blue region is excluded by WMAP 7 yrs bound on tensors $r < 0.24$. The black diagonal line represents our bound Eq. (6). The magenta diagonal line represents the Efstathiou-Mack relationship [6] describing models based on the potential $V(\phi) = V_0 - \lambda\phi^4/4$. Regions above these lines are excluded. The black vertical line corresponds to $\Delta\phi = M_P$. The horizontal dashed lines represent the forecasted sensitivity of forthcoming observations. All the models are plotted for $N = 60$.

Actually, Eq. (10) is a special case of a more general relationship

$$\frac{\Delta\phi}{m_P} = r^{\frac{p-2}{2p}} \frac{[p(p-2)N]^{1/p}}{\sqrt{8\pi}} \left(\frac{N(p-2)}{2\sqrt{2}} \right)^{\frac{p-2}{p}} \quad (11)$$

holding for general hilltop models described by the potential $V(\phi) = V_0 [1 - \lambda_p(\phi/\mu)^p]$, where $p > 2$ and $M_P > \mu > 0$ regardless of the values of the parameters of the potential i.e. V_0 , μ and λ_p . Therefore, there is a whole family of models that satisfies this relationship. Next, let us consider chaotic models which are characterized by a power law potential $V(\phi) \propto \phi^p$. As hilltop models they have the property that ϵ increases monotonically during inflation. Therefore, the bound on the tensor-to-scalar ratio Eq. (6) still holds as in the case of hilltop inflation. However, replacing $\phi_* \simeq \sqrt{2pN}M_P$, the resulting bound

$$r \lesssim 0.24p(60/N) . \quad (12)$$

is hardly constraining even for $p = 1$.

What about Natural inflation? Natural inflation is the only known field-theoretically consistent implementation of chaotic inflation. The inflaton is a PNCB, with

symmetry breaking scale $f \gg M_P$. In general the potential can be written as $V(\phi) \propto [1 + \cos(\phi/f)]$, where $f \equiv M_P/\sqrt{2\eta_0}$, $\eta_0 > 0$ is the second slow-roll parameter at the top of the potential and it reduces to a quadratic hilltop potential for small angles. For $N = 60$, using the WMAP 7yrs bound on n_s , we can constrain $\eta_0 \lesssim 0.017$. This in turn gives a range of tensor fraction $0.04 \lesssim r \lesssim 0.13$ that is compatible with the later bound.

Let us now consider de Sitter entropy bounds. The second law of thermodynamics states that the entropy of any closed system never decreases with time. The expanding universe during slow-roll inflation does not escape this rule [7]. The entropy of de Sitter spacetime is given by $S_{dS} = 8\pi^2 M_P^2/H^2$, and it varies during N e-folds of slow-roll as

$$\frac{dS_{dS}}{dN} = 16\pi^2 \frac{M_P^2}{H^2} \epsilon = 2\langle\zeta^2\rangle^{-1}, \quad (13)$$

where we used $\langle\zeta^2\rangle \equiv H^2/8\pi^2 M_P^2 \epsilon$. It is easy to show that the integral of Eq. (13) is bounded as follows

$$\langle\zeta_{\max}^2\rangle^{-1} \leq \Delta S_{dS}/2N \leq \langle\zeta_{\min}^2\rangle^{-1}. \quad (14)$$

This inequality is valid independently the variation of the

slow-roll parameter in the non-eternal inflation regime. Neglecting the Hubble parameter variation during inflation and taking $\epsilon_{\min} = \epsilon_c$ for non-eternal inflation the left-hand side of Eq. (14) yields the well-known bound on e-folds from de Sitter entropy [8] $N \leq S_{dS}/12$. On the other hand, the right-hand side of Eq. (14) leads to the trivial and model-independent lower bound on the number of e-folds $N \geq 1/2$. Now, specializing to the typical case of monotonically increasing ϵ , i.e. $\epsilon_{\min} = \epsilon_*$, the left-hand side of Eq. (14) yields $N \leq \langle \zeta_*^2 \rangle \Delta S_{dS}/2$, which in turn can be written as a general model-independent bound on the tensor fraction

$$r \lesssim 0.13 (60/N) (H_*/H_{\text{end}})^2. \quad (15)$$

This is the second main result of this paper. At first sight, Eq. (15) appears to be a stringent bound, if it were not for the prefactor $(H_*/H_{\text{end}})^2$ that degrades it for models where the Hubble rate changes appreciably. This is the case for instance for chaotic models. Nevertheless, this is an absolute bound on the amount of produced tensor modes $r \lesssim 0.13$ for scenarios where $H \simeq \text{constant}$ during inflation.

Let us turn to discuss the observational situation which is really promising. The *Planck* mission [9] is aiming to reach $r \lesssim 0.1$ in which case the simplest quadratic chaotic inflation scenario will be probed. In addition, a variety of experimental setups [10–12] is targeting the range $10^{-3} \lesssim r \lesssim 0.1$. The lowest detectable tensor fraction through CMB polarization is probably $r \simeq 10^{-4}$ both from polarized dust foregrounds subtraction [13] and contamination from E to B -modes conversion through lensing [14]. In Fig. 1, we represent the various models in the plane $\Delta\phi$ versus r , instead of the traditional r versus n_s plot, together with our bound Eq. (6), and experimental reaches of the planned observations. We did not represent the family of models that satisfy Eq. (11) as they lay closely to the magenta line representing the Efstathiou-Mack relationship. As expected, Natural inflation appears to interpolate between quadratic hilltop and quadratic chaotic models. It is also noteworthy, though well-known, that none of the single-field scenarios compatible with observation has a sub-Planckian inflaton excursion.

In conclusion, we reiterate the truism that the tensor fraction is a very important observable to test the initial conditions of the universe. We derived two theoretical bounds on that quantity. The first bound Eq. (6) means that (for sub-Planckian inflation excursion and thus field-theoretically sound description) $r \lesssim 0.002$, placing it beyond the reach of *Planck* but within reach of *CoRE* and *PIXIE* [12]. On the other hand, the fact that both single-field benchmark scenarios that are consistent with current data have $\Delta\phi \gtrsim 10 M_P$ make their tensor fraction within reach of *CMBPol*. The second bound, arising from de Sitter entropy bounds implies an upper bound on tensors Eq. (15), independently of the magnitude of

inflaton excursion, which is within reach of *Planck* and future observations. The detection of B -modes would promote inflation from an attractive paradigm to a predictive theory. However this is not the end of the story as any realization of slow-roll inflation will have to face the issue of a consistent UV-completion.

We end up with some speculations. It would be interesting to explore the relationship between the entropy of the inflaton and r . For instance, in [15] it was argued that the entropy of the inflaton is proportional to $(\Delta\phi/H)^2$, which by the de Sitter entropy bound, would censor any attempt to have Super-Planckian inflaton excursion, and thus observable inflationary GWs, in a consistent UV-complete theory. This is in agreement with the conclusions reached in [16], where it was shown that attempts to build natural inflation models in string theory with decay constant $f \gg M_P$ are doomed to failure.

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